European Society of Computational Methods in Sciences and Engineering (ESCMSE)

Journal of Numerical Analysis, Industrial and Applied Mathematics (JNAIAM)
vol. 9-10, no. 1-2, 2015, pp. 1-10
ISSN 1790–8140

Rate of convergence estimates for second order elliptic eigenvalue problems on polygonal domains using spectral element methods

Lokendra K. Balyan[1], **Subir Singh Lamba**
Department of Mathematics, IIIT-DM Jabalpur, India

Received 7 January 2014; accepted in revised form 15 July 2015

Abstract: In this paper, we present rate of convergence estimates for eigenvalues and eigenvectors of elliptic differential operators on non-smooth domains using non-conforming spectral element methods. We define a class of compact operators on Banach space which is used to obtain the results. If coefficients of the differential operator are sufficiently smooth and the boundaries of the polygonal domain are piecewise analytic then exponential convergence to approximate solution is obtained.

Keywords: Rate of convergence; eigenvalues and eigenvectors; non-smooth domains; singularities; *h-p* spectral element method.

Mathematics Subject Classification: 35Jxx; 35Pxx

1 Introduction

In this paper, we present convergence estimates for eigenvalues and eigenvectors of elliptic differential operators using spectral element methods. In the era of computation the spectral methods have been proven to be faster and more accurate as compared to other alternative methods like finite element methods and finite difference methods [7, 9, 14], and the references therein. The method provides exponential convergence if the solution is sufficiently smooth, which would normally be lost when the solution shows singular behaviour at the corners of non-smooth domains.

Spectral methods for solving the problems on non-smooth domains allow only algebraic convergence [9, 14] which can be improved by use of auxiliary mapping of the form $z = log\, \xi$, as proposed by Kondratiev [15]. Babuska and Guo used geometrically fine geometrical mesh in the neighborhood of each of the corner in the framwork of finite element method [2, 3, 4]. To overcome the singularities at corners we also use similar kind of geometrically fine geometrical mesh and auxiliary mapping for obtaining exponential accurate solutions. Spectral methods for solving elliptic boundary value problems on non smooth domains proposed in [13, 18]. The author et al. have obtained the solution of elliptic eigenvalue problems on non smooth domains using spectral element method [6].

[1]Corresponding author: E-mail: lokendra.balyan@gmail.com

In [8] Bramble and Osborn developed spectral approximation results for a class of compact operators on a Hilbert space and used the operators to obtain rate of convergence estimates for the approximation of eigenvalues and generalized eigenvectors of non-selfadjoint elliptic differential operators using Galerkin type methods. Later in [16] Osborn presented the spectral approximation results for compact operators on a Banach space. The author claimed the use of these results for various kind of approximation like, approximation of integral operators by numerical quadrature, Galerkin approximation for non-selfadjoint elliptic differential operators and formulated the results for approximation of eigenvalues and generalized eigenvectors of elliptic eigenvalue problems in terms of the norm on the underlying Banach spaces. Descloux et al. in [11, 12] developed the convergence estimates for non compact operators on Banach Spaces which were approximated by Galerkin method.

In our work, the rate of convergence of eigenvalues is reduced to the problem of rate of convergence of the approximate solution of corresponding elliptic boundary value problems as [5, 8]. Similar to [16] we define a compact operator $\mathcal{T} : V \to V, V$ is a Banach space, and a family of compact operators $\mathcal{T}^P : V \to V$, such that $\mathcal{T}^P \to \mathcal{T}$ in H^1 norm, as $P \to \infty$. Here P denotes the degree of polynomials. We obtain the convergence estimates of $\mathcal{T}^P$, from $\mathcal{T}$ using spectral element method. Babuska et.al [5] have shown that a compact operator $\mathcal{T}$ satisfies the variational formulation $a(\mathcal{T}u, v) = b(u, v)$ corresponding to elliptic boundary value problems. Let λ be a nonzero eigenvalue of $\mathcal{T}$ which satisfies variational formulation $a(u, v) = \lambda b(u, v)$, u is called an associate eigenvector, corresponding to eigenvalue problems. Similarly we concluded that (λ, u) is an eigenpair of eigenvalue problem if and only if $\lambda\mathcal{T}u = u$, i.e. (λ^{-1}, u) is an eigenpair of the compact operator $\mathcal{T}$ [5, 10]. Further we define an integer ι which satisfy $\mathcal{N}((\lambda - \mathcal{T})^\iota) = \mathcal{N}((\lambda - \mathcal{T})^{\iota+1})$, where $\mathcal{N}$ denotes the null space, is called the ascent of $\lambda - \mathcal{T}$. The vectors in $\mathcal{N}((\lambda - \mathcal{T})^\iota)$ are called the generalized eigenvectors of $\mathcal{T}$ corresponding to λ [5, 8, 16].

The outline of the paper is as follows. In section 2, we define our problem on non-smooth domain and discuss the discretization of the domain. In section 3, we present our main result which shows exponential convergence for the approximation of eigenfunctions.

2 Discretization

In this section we explain the methodology and discretization of domain for elliptic boundary value problems. Consider the boundary value problem

$$\begin{aligned} \mathcal{L}u &= f \text{ in } \Omega, \\ u &= g^{[0]} \text{ on } \Gamma^{[0]}, \\ \left(\frac{\partial u}{\partial N}\right)_{\mathcal{A}} &= g^{[1]} \text{ on } \Gamma^{[1]}. \end{aligned} \tag{1}$$

Here $\mathcal{L}$ is a strongly elliptic operator defined as

$$\mathcal{L}u = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{2}$$

where the coefficients $a_{i,j}(x) = a_{j,i}(x)$, $b_i(x)$ and $c(x)$ are analytic functions of x. Here $\left(\frac{\partial u}{\partial N}\right)_{\mathcal{A}}$ is the conormal derivative of u. Let $\mathcal{A}$ denote the 2×2 matrix whose entries are given by

$$\mathcal{A}_{i,j}(x) = a_{i,j}(x),$$

for $i, j = 1, 2$. Then $\left(\frac{\partial u}{\partial N}\right)_{\mathcal{A}}$ is defined as

$$\left(\frac{\partial u}{\partial N}\right)_{\mathcal{A}} = \sum_{i,j=1}^{2} a_{i,j} n_j \frac{\partial u}{\partial x_i}, \tag{3}$$

where $n(x) = (n_1, n_2)$ is the exterior unit normal to Γ at x. Here Γ denotes the boundary of the domain. Further, let $\Gamma = \Gamma^{[0]} \cup \Gamma^{[1]}$, $\Gamma^{[0]} = \cup_{i \in \mathcal{D}} \bar{\Gamma}_i$ and $\Gamma^{[1]} = \cup_{i \in \mathcal{N}} \bar{\Gamma}_i$. $\Gamma^{[0]}$ denotes the Dirichlet boundary and $\Gamma^{[1]}$ the Neumann boundary.

Let Ω be a polygon with vertices $A_1, ..., A_p$ and corresponding sides $\Gamma_1, ..., \Gamma_p$, where Γ_i joins the points A_{i-1} and A_i. We divide Ω into p sub domains $S^1, S^2, ..., S^p$ such that each S^k contains

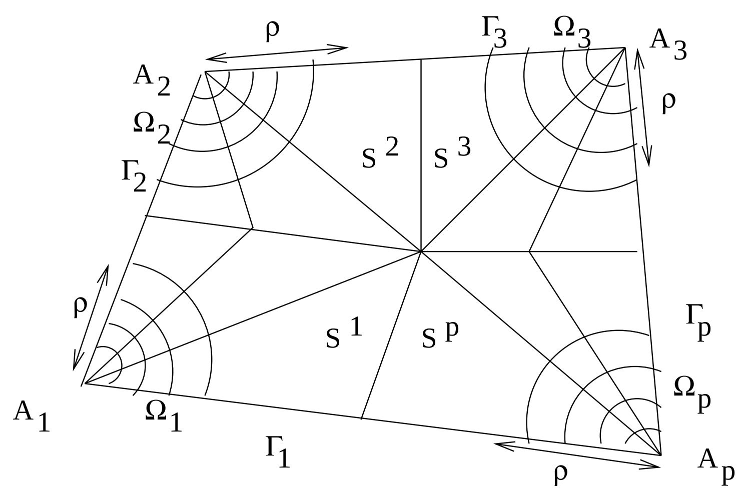

Figure 1: Discretization of the domain Ω into p non overlapping subdomains.

only the singularities at the vertex A_k, see Fig. 1. Then the sector S^k with center at A_k is defined by

$$S^k = \{ \, (x, y) \mid 0 < r_k < \rho, \psi^k_{\min} < \theta_k < \psi^k_{\max} \, \}. \tag{4}$$

Here (r_k, θ_k) are polar coordinates at vertex A_k and ρ is chosen such that $S^k \cap S^l = \emptyset$ for $k \neq l$. $\{\psi^k_i \mid i = 1, ..., I_{k+1}\}$ is an increasing sequence of points such that $\psi^k_1 = \psi^k_{\min}$ and $\psi^k_{I_{k+1}} = \psi^k_{\max}$.

Choose $\mu = (\mu_1, ..., \mu_p)$ to be the geometric ratios with $0 < \mu_i < 1$. Let

$$\sigma^k_1 = 0 \text{ and } \sigma^k_j = \rho(\mu_k)^{N+1-j} \text{ for } 2 \leq j \leq N+1,$$

and

$$\Omega^k_{i,j} = \{ \, (r_k, \theta_k) \mid \sigma^k_j < r_k < \sigma^k_{j+1}, \psi^k_i < \theta_k < \psi^k_{i+1} \, \}, \tag{5}$$

for $1 \leq i \leq I_k, 1 \leq j \leq N$.

Next, we define $\eta^k_j = ln \, \sigma^k_j$ for $0 \leq j \leq N+1$. Here $\eta^k_0 = -\infty$. Now we shall denote the image of $\Omega^k_{i,j}$ in (τ_k, θ_k) coordinates by $\tilde{\Omega}^k_{i,j}$, where

$$\tilde{\Omega}^k_{i,j} = \{ \, (\tau_k, \theta_k) \mid \eta^k_j < \tau_k < \eta^k_{j+1}, \psi^k_i < \theta_k < \psi^k_{i+1} \, \}, \tag{6}$$

for $1 \leq i \leq I_k$, $0 \leq j \leq N$. In doing so the geometric mesh $\Omega^k_{i,j}$ for $1 \leq j \leq N$ reduces to a quasi uniform mesh. However $\tilde{\Omega}^k_{i,0}$ is a semi-infinite strip.

In the remaining part of S^k, we retain the Cartesian coordinate system (x, y). Let

$$O^{p+1} = \{\Omega^k_{i,j} \mid 1 \leq k \leq p, N < j \leq J_k, 1 \leq i \leq I_k\}.$$

We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \{\Omega^{p+1}_l \mid 1 \leq l \leq L\}, \tag{7}$$

where L denotes the cardinality of O^{p+1}. Henceforth the elements in O^{p+1} are chosen to be rectangles.

We shall now define the set of non-conforming spectral element functions in a general monomial form over the sectoral neighbourhood in (τ_k, θ_k) coordinates, where $\tau_k = \ln r_k$. However away from the corners the spectral element functions are defined in (ξ, η) coordinates [6, 13]. Let $\{\{\hat{u}^k_{i,j}(\tau_k, \theta_k)\}_{i,j,k}\} \in \Pi^{N,P}$ be spectral element functions defined, by assuming that the cornermost element is constant, i.e. $\hat{u}^k_{i,1} = c_k$ for all $1 \le i \le I_k$ and

$$\hat{u}^k_{i,j} = \sum_{r=0}^{P_j}\sum_{s=0}^{P_j} b_{r,s}\tau_k^r\theta_k^s,$$

on $\hat{\Omega}^k_{i,j}$ for $1 \le i \le I_k, 2 \le j \le N, 1 \le k \le p$. Here $1 \le P_j \le P$.
Moreover, there is an analytic mapping M_l^{p+1} from the master square $S = (-1,1)^2$ to Ω_l^{p+1} and hence spectral element function is define by

$$\hat{u}_l^{p+1}(\xi, \eta) = \sum_{r=0}^{P}\sum_{s=0}^{P} b_{r,s}\xi^r\eta^s.$$

Let $d\,(A_k, \gamma_s) = \inf_{x\in\gamma_s}\{\text{distance}(A_k, x)\}$ denote the distance between x and vertex A_k. Let $\beta = (\beta_1, \beta_2, ..., \beta_p)$ denote a p-tuple of real numbers, $0 < \beta_i < 1, i = 1, ..., p$.

Let $f \in H^{1,0}_\beta(\Omega)$ and we choose $\beta_k^{'}$ so that $\beta_k^* < \beta_k^{'} < \beta_k$ then by shift theorem [1] the solution of (1) exists in $H^{3,2}_{\beta^{'}}(\Omega)$. Further by using the Remark 3 of [2] we can obtained the estimates

$$\|u\|_{H^{3,2}_{\beta^{'}}(\Omega)} \le C\|f\|_{H^{1,0}_{\beta^{'}}(\Omega)}. \tag{8}$$

We now define the quadratic form which is needed in the sequel.

$$\begin{aligned}
&\mathcal{V}^{N,P}_{vertices}\left(\{\hat{u}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}\right)\\
&= \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\alpha_k}\left\|\hat{\mathcal{L}}^k\hat{u}^k_{i,j}(\tau_k,\theta_k)\right\|^2_{0,\hat{\Omega}^k_{i,j}}\\
&+\sum_{k=1}^{p}\sum_{\substack{\gamma_s\subseteq\Omega^k\cup B^k_\rho,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left(\left\|[\hat{u}^k]\right\|^2_{0,\hat{\gamma}_s}+\left\|[\hat{u}^k_{\tau_k}]\right\|^2_{\frac{1}{2},\hat{\gamma}_s}+\left\|[\hat{u}^k_{\theta_k}]\right\|^2_{\frac{1}{2},\hat{\gamma}_s}\right)\\
&+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\left(|\,c_k\,|^2+\sum_{\substack{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left(\left\|\hat{u}^k-c_k\right\|^2_{0,\hat{\gamma}_s}+\left\|\hat{u}^k_{\tau_k}\right\|^2_{\frac{1}{2},\hat{\gamma}_s}\right)\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\substack{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left\|\left(\frac{\partial\hat{u}^k}{\partial n}\right)_{\hat{\mathcal{A}}_k}\right\|^2_{\frac{1}{2},\hat{\gamma}_s}.
\end{aligned}\tag{9}$$

Here $\mu(\hat{\gamma}_s)$ is the length of the curve $\hat{\gamma}_s$ and $\alpha_k = 1 - \beta_k^*$.

Next, we define

$$\begin{aligned}
&\mathcal{V}^{N,P}_{interior}\left(\{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right)\\
&=\sum_{l=1}^{L}\left\|\mathcal{L}^{p+1}_l\hat{u}^{p+1}_l\right\|^2_{0,S}+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|\left[u^{p+1}\right]\right\|^2_{0,\gamma_s}+\left\|\left[u^{p+1}_{x_1}\right]\right\|^2_{\frac{1}{2},\gamma_s}+\left\|\left[u^{p+1}_{x_2}\right]\right\|^2_{\frac{1}{2},\gamma_s}\right)\\
&+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\left\|u^{p+1}\right\|^2_{0,\gamma_s}+\left\|\frac{\partial u^{p+1}}{\partial t}\right\|^2_{\frac{1}{2},\gamma_s}\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left\|\left(\frac{\partial u^{p+1}}{\partial n}\right)_{\mathcal{A}}\right\|^2_{\frac{1}{2},\gamma_s}. \qquad (10)
\end{aligned}$$

Let

$$\begin{aligned}
\mathcal{V}^{N,P}\left(\{\hat{u}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k},\{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right) &= \mathcal{V}^{N,P}_{vertices}\left(\{\hat{u}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}\right)\\
&+\mathcal{V}^{N,P}_{interior}\left(\{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right). \qquad (11)
\end{aligned}$$

All the required terms in above definition are widely explained in [6].

3 Convergence estimates

In this section we shall obtain estimates for the error in approximating the eigenvalues and eigenvectors which is similar to the proof of error estimates in [6, 18].

Let $\mathcal{T}$ be a compact operator and $\mathcal{T}^P$ be a family of compact operators. Then we can relate the projection onto the generalized null space corresponding to eigenvalue λ to the projection associated with part of the spectrum of $\mathcal{T}^P$. Let λ be a nonzero eigenvalue of $\mathcal{T}$ with algebraic multiplicity m and Γ be a circle centered at λ which lies in $\rho(\mathcal{T})$ and contains no other points of $\sigma(\mathcal{T})$. Then the spectral projection associated with $(\lambda,\mathcal{T})$ is defined by

$$E=E(\lambda)=\frac{1}{2\pi i}\int_\Gamma R_z(T)dz. \qquad (12)$$

Here E is a projection onto the space of generalized eigenvectors associated with λ and $\mathcal{T}$, i.e., $R(E)=\mathcal{N}((\lambda-T)^\iota)$, where $R(E)$ denotes the range of E [10].

Theorem 3.1 Let $f\in R(E)$. Then there exist constants a and k such that

$$\|\left(\mathcal{T}-\mathcal{T}^P\right)f\|\leq a\,e^{-k\,P}. \qquad (13)$$

Proof: Let $f\in R(E)$ and $u=\mathcal{T}(f)$. Then $\mathcal{T}$ is compact operator [14],[17]. Let $\hat{U}^k_{i,j}(\tau_k,\theta_k)=u(x_1,x_2)-c_k$ for $(\tau_k,\theta_k)\in\tilde{\Omega}^k_{i,j}$, and $c_k=u(A_k)$. Then by using *Proposition* 2.1 of [13] the estimate

$$\begin{aligned}
&\int_{\eta^k_j}^{\eta^k_{j+1}}\int_{\psi^k_j}^{\psi^k_{j+1}}\sum_{|\varepsilon|\leq m}|D^{\varepsilon_1}_{\tau_k}D^{\varepsilon_2}_{\theta_k}(\hat{U}^k_{i,j})|^2e^{-2\lambda_k\tau_k}d\tau_k\,d\theta_k\\
&\leq C(\rho\mu_k^{N+1-j})^{2\gamma_k}(Cd^{m-2}(m-2)!)^2 \qquad (14)
\end{aligned}$$

holds for $1\leq j\leq N$, where $0<\lambda_k<\alpha_k$ and $\gamma_k<\alpha_k-\lambda_k$. Here C and d are constants independent of m.

Now consider a quadrilateral $\Omega_l^{p+1} \in \Omega^{p+1}$. Let

$$\hat{U}_l^{p+1}(\xi,\eta) = u\left(M_l^{p+1}(\xi,\eta)\right).$$

Then the following has been proved in *Lemma* 5.1 of [4].
$\hat{U}_l^{p+1}(\xi,\eta)$ is analytic on $\bar{S}$ and hence for some constants C and d, we have

$$\mid D^\alpha \hat{U}_l^{p+1}(\xi,\eta) \mid \leq C\, d^m m! \tag{15}$$

for $|\alpha| = m$, $m = 1, 2, ...$
Now using the results on approximation theory in [4, 17] there exists a polynomial $\hat{\Phi}_l^{p+1}(\xi,\eta)$ of degree P in each variable separately such that

$$\|\hat{U}_l^{p+1}(\xi,\eta) - \hat{\Phi}_l^{p+1}(\xi,\eta)\|_{2,S}^2 \ \leq\ C_s P^{-2s+8}(Cd^s s!)^2 \tag{16}$$

for $1 \leq l \leq L$, where $C_s = Ce^{2s}$.
Define $\hat{U}_{i,j}^k(\tau_k,\theta_k) = u(x_1,x_2) - c_k$ for $(x_1,x_2) \in \Omega_{i,j}^k$, where $c_k = u(A_k)$. Then there exists a polynomial $\hat{\Phi}_{i,j}^k(\tau_k,\theta_k)$ of degree P_j in τ_k and θ_k separately such that

$$\|\hat{U}_{i,j}^k(\tau_k,\theta_k) - \hat{\Phi}_{i,j}^k(\tau_k,\theta_k)\|_{2,\tilde{\Omega}_{i,j}^k}^2 \ \leq\ C_{s_j} P_j^{-2s_j+8}(\chi_k)^{2s_j}\, \|\hat{U}_{i,j}^k\|_{s_j,\tilde{\Omega}_{i,j}^k}^2 \tag{17}$$

where

$$\chi_k = \max\left\{\frac{1}{2}\max_i(\Delta\psi_i^k), \frac{\mid ln\mu_k \mid}{2}, 1\right\}. \tag{18}$$

We now define the functional which is to be minimized.

$$\begin{aligned}
&\mathcal{R}_{vertices}^{N,P}\left(\{\hat{u}_{i,j}^k(\tau_k,\theta_k)\}_{i,j,k}\right)\\
&= \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\alpha_k}\left\|\hat{\mathcal{L}}^k\hat{u}_{i,j}^k(\tau_k,\theta_k) - F^k(\tau_k,\theta_k)\right\|_{0,\hat{\Omega}_{i,j}^k}^2\\
&+\sum_{k=1}^{p}\sum_{\substack{\gamma_s\subseteq\Omega^k\cup B_\rho^k,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left(\left\|[\hat{u}^k]\right\|_{0,\hat{\gamma}_s}^2 + \left\|[\hat{u}_{\tau_k}^k]\right\|_{\frac{1}{2},\hat{\gamma}_s}^2 + \left\|[\hat{u}_{\theta_k}^k]\right\|_{\frac{1}{2},\hat{\gamma}_s}^2\right)\\
&+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\left(\mid c_k\mid^2 + \sum_{\substack{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left(\left\|\hat{u}^k - c_k\right\|_{0,\hat{\gamma}_s}^2 + \left\|\hat{u}_{\tau_k}^k\right\|_{\frac{1}{2},\hat{\gamma}_s}^2\right)\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\substack{\gamma_s\subseteq\partial\Omega^k\cap\Gamma_l,\\ \mu(\hat{\gamma}_s)<\infty}} d(A_k,\gamma_s)^{-2\alpha_k}\left\|\left(\frac{\partial\hat{u}^k}{\partial n}\right)_{\hat{A}_k}\right\|_{\frac{1}{2},\hat{\gamma}_s}^2.
\end{aligned} \tag{19}$$

Next, we define

$$\begin{aligned}
&\mathcal{R}^{N,P}_{interior}\left(\{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right) \\
&= \sum_{l=1}^{L}\left\|\mathcal{L}^{p+1}_l\hat{u}^{p+1}_l - F^{p+1}_l(\xi,\eta)\right\|^2_{0,S} \\
&+ \sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|[u^{p+1}]\right\|^2_{0,\gamma_s} + \left\|[u^{p+1}_{x_1}]\right\|^2_{\frac{1}{2},\gamma_s} + \left\|[u^{p+1}_{x_2}]\right\|^2_{\frac{1}{2},\gamma_s}\right) \\
&+ \sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\left\|u^{p+1}\right\|^2_{0,\gamma_s} + \left\|\frac{\partial u^{p+1}}{\partial t}\right\|^2_{\frac{1}{2},\gamma_s}\right) \\
&+ \sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left\|\left(\frac{\partial u^{p+1}}{\partial n}\right)_{\mathcal{A}}\right\|^2_{\frac{1}{2},\gamma_s}.
\end{aligned} \tag{20}$$

Let

$$\begin{aligned}
\mathcal{R}^{N,P}\left(\{\hat{u}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right) &= \mathcal{R}^{N,P}_{vertices}\left(\{\hat{u}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}\right) \\
&+\mathcal{R}^{N,P}_{interior}\left(\{\hat{u}^{p+1}_l(\xi,\eta)\}_l\right).
\end{aligned}$$

Define $\hat{\Psi}^k_{i,0}(\tau_k,\theta_k) = c_k$ for $1\leq i\leq I_k, 1\leq k\leq p$, and $\hat{\Psi}^k_{i,j}(\tau_k,\theta_k) = \hat{\Phi}^k_{i,j}(\tau_k,\theta_k) + c_k$ for $(\tau_k,\theta_k)\in\tilde{\Omega}^k_{i,j}$ and $\hat{\Psi}^{p+1}_l(\xi,\eta) = \hat{\Phi}^{p+1}_l(\xi,\eta)$ for $(\xi,\eta)\in S$.
Now

$$\left\|\tilde{\mathcal{L}}^k\hat{\Psi}^k_{i,j} - F^k_{i,j}\right\|^2_{0,\tilde{\Omega}^k_{i,j}} \leq \left\|\tilde{\mathcal{L}}^k(\hat{\Psi}^k_{i,j}+c_k) - \tilde{\mathcal{L}}^k(\hat{U}^k_{i,j}+c_k)\right\|^2_{0,\tilde{\Omega}^k_{i,j}} \tag{21}$$

Hence

$$\begin{aligned}
&(\rho\mu_k^{N+1-j})^{-2\alpha_k}\left\|\tilde{\mathcal{L}}^k\hat{\Psi}^k_{i,j} - F^k_{i,j}\right\|^2_{0,\tilde{\Omega}^k_{i,j}} \\
\leq\ &(\rho\mu_k^{N+1-j})^{-2\alpha_k}\left\|\tilde{\mathcal{L}}^k(\hat{\Phi}^k_{i,j}+c_k) - \tilde{\mathcal{L}}^k(\hat{U}^k_{i,j}+c_k)\right\|^2_{0,\tilde{\Omega}^k_{i,j}} \\
\leq\ &C(\rho\mu_k^{N+1-j})^{-2\alpha_k}\left\|\hat{\Phi}^k_{i,j} - \hat{U}^k_{i,j}\right\|^2_{2,\tilde{\Omega}^k_{i,j}} \\
\leq\ &(\rho\mu_k^{N+1-j})^{-2\alpha_k}C_{s_j}(P_j)^{-2s_j+8}(\chi_k)^{2s_j}\left\|\hat{U}^k_{i,j}\right\|^2_{2,\tilde{\Omega}^k_{i,j}} \\
\leq\ &C_{s_j}(P_j)^{-2s_j+8}\left(C(\rho\mu_k^{N+1-j})^{\gamma_k}(\chi_k d)^{s_j-2}(s_j-2)!\right)^2. \qquad \text{(by 14)}
\end{aligned}$$

We proceed all the other terms of $\mathcal{R}^{N,P}_{exterior}\left(\left\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}\right)$ in same fashion and finally estimate

$$\mathcal{R}^{N,P}_{exterior}\left(\left\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}\right) \leq \sum_{k=1}^{p}\sum_{j=1}^{N}C_{s_j}(P_j)^{-2s_j+8}\left(C(\rho\mu_k^{N+1-j})^{\gamma_k}(\chi_k d)^{s_j}s_j!\right)^2. \tag{22}$$

Similarly it can be shown that

$$\mathcal{R}^{N,P}_{interior}\left(\left\{\hat{\Psi}^{p+1}_l(\xi,\eta)\right\}_l\right) \leq \left(C_s(P)^{-2s+8}(Cd^s s!)^2\right). \tag{23}$$

We choose

$$\alpha j \leq P_j \leq \beta P, \quad \text{where } 0 < \alpha \text{ and } \beta \leq 1, \tag{24}$$

$$s_j \leq \Upsilon P_j, \quad \text{where } 0 < \Upsilon < 1. \tag{25}$$

Moreover $s = \Upsilon P$.
Now by using Stirling's formula

$$n! \sim \sqrt{2\pi n}\, e^{-n} n^n,$$

we obtain

$$\begin{aligned} &\mathcal{R}^{N,P}(\left\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\hat{\Psi}^{p+1}_l(\xi,\eta)\right\}_l) \\ &\leq \sum_{k=1}^{p}\sum_{j=1}^{N} C(P_j)^8(\rho\mu_k^{N+1-j})^{2\gamma_k}\left(2\pi\Upsilon P_j(\chi_k\Upsilon d)^{2\Upsilon P_j}\right) \\ &+ C\left(P^8(2\pi\Upsilon P)(\Upsilon d)^{2\Upsilon P}\right). \end{aligned}$$

Select Υ so that $(\chi_k\Upsilon d) < 1$ for all k and $\Upsilon d < 1$. Then there exists a constant $b > 0$ such that the estimate

$$\mathcal{R}^{N,P}\left(\left\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\hat{\Psi}^{p+1}_l(\xi,\eta)\right\}_l\right) \leq Ce^{-bP} \tag{26}$$

holds.

Let $\left(\{\hat{w}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{w}^{p+1}_l(\xi,\eta)\}_l\right)$ be the space of spectral element functions, which minimizes the functional $\mathcal{R}^{N,P}\left(\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{\Psi}^{p+1}_l(\xi,\eta)\}_l\right)$ over all i,j,k.
Hence

$$\mathcal{R}^{N,P}\left(\left\{\hat{w}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\hat{w}^{p+1}_l(\xi,\eta)\right\}_l\right) \leq Ce^{-bP}. \tag{27}$$

Therefore we can conclude that

$$\mathcal{V}^{N,P}\left(\left\{\hat{\Psi}^k_{i,j}(\tau_k,\theta_k) - \hat{w}^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\hat{\Psi}^{p+1}_l(\xi,\eta) - \hat{w}^{p+1}_l(\xi,\eta)\right\}_l\right) \leq Ce^{-bP} \tag{28}$$

Hence by using the Stability Theorem 3.1 of [13] it can be shown that there exist constants a and k such that

$$\begin{aligned} &\sum_{k=1}^{p}\left(|\, c_k - h_k\,|^2 + \sum_{j=1}^{N}\sum_{i=1}^{I_k}(\rho\mu_k^{N+1-j})^{-2\lambda_k}\|(\hat{w}^k_{i,j} - \hat{U}^k_{i,j})(\tau_k,\theta_k) - (h_k - a_k)\|^2_{2,\tilde{\Omega}^k_{i,j}}\right) \\ &+ \sum_{l=1}^{L}\|(\hat{w}^{p+1}_l - \hat{U}^{p+1}_l)(\xi,\eta)\|^2_{2,S} \leq ae^{-k\,P} \end{aligned} \tag{29}$$

holds.

Now we construct a set of corrections $\left(\{\hat{\alpha}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{\alpha}^{p+1}_l(\xi,\eta)\}_l\right) \in \Pi^{N,P}$ such that

$$\begin{aligned} \left(\{\hat{r}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{r}^{p+1}_l(\xi,\eta)\}_l\right) &= \left(\{\hat{w}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{w}^{p+1}_l(\xi,\eta)\}_l\right) \\ &+ \left(\{\hat{\alpha}^k_{i,j}(\tau_k,\theta_k)\}_{i,j,k}, \{\hat{\alpha}^{p+1}_l(\xi,\eta)\}_l\right) \end{aligned}$$

is conforming and belongs to $H^1(\Omega)$ [6].

Define the operator $\mathcal{T}^P(f) = r$. Then $\mathcal{T}^P$ is a compact operator since its range is finite dimensional. Then the error estimate

$$\|(\mathcal{T} - \mathcal{T}^P)(f)\|_{1,\Omega} \leq ae^{-k\,P}$$

holds for P large enough.

4 Conclusion

In this paper, we defined a class of compact operators and used them to obtain the exponential convergence for the approximation of eigenvalues and generalized eigenvectors of second order elliptic differential operator on non-smooth domains, where the solution exhibit the singular behaviour at corners, using non-conforming approach of spectral element methods.

Acknowledgments

The authors wish to thank Professor Pravir Dutt, Department of Mathematics and Statistics, IIT Kanpur for their help and guidance for the work. The authors would also like to thank unknown referees for their valuable comments and suggestions for improvement of the paper.

References

[1] R.A. Adams: Sobolev spaces, Academic Press (1975).

[2] I. Babuska and B.Q. Guo: Regularity of the solution of elliptic problems with piecewise analytic data, Part-I. SIAM J. Math. Anal. 19, 172-203 (1988).

[3] I. Babuska and B.Q. Guo: Regularity of the solution of elliptic problems with piecewise analytic data, Part-II. SIAM J. Math. Anal. 20, 763-781 (1989).

[4] I. Babuska and B.Q. Guo: The *h-p* version of the finite element method on domains with curved boundaries. SIAM J. Num. Anal. 25(4), 837-861 (1988) .

[5] I. Babuska and J. Osborn: Eigenvalue problems, Handbook of numerical analysis, V.2, Finite element methods (part II). Edited by P.G. Ciarlet and J.L. Lions (1991) .

[6] L.K. Balyan, P. Dutt and R.K.S. Rathore: Least squares *h-p* spectral element method for elliptic eigenvalue problems. Applied Mathematics and Computation Vol 218(19), 9596-9613 (2012).

[7] J.P. Boyd: Chebyshev and Fourier spectral methods, second edition, DOVER publ. Mineola, New York (2000).

[8] J.H. Bramble and J.E. Osborn: Rate of convergence estimates for non-selfadjoint eigenvalue approximation. Math. Comp. 27, 525-549 (1973).

[9] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang: Spectral methods in fluid dynamics, Series in Computational Physics, Springer Verlag (1988).

[10] F. Chatelin: Spectral approximations of linear operators. Academic Press, New York (1983).

[11] J. Descloux, N. Nassif and J. Rappaz: On spectral approximation , Part 1: The problems of convergence, RAIRO Anal. Numer. 12, 97-112, (1978).

[12] J. Descloux, N. Nassif and J. Rappaz: On spectral approximation , Part 2: Error estimates for the Galerkin method, RAIRO Anal. Numer. 12, 113-119, (1978).

[13] P. Dutt, N. Kishore and C.S. Upadhyay: Nonconforming *h-p* spectral element methods for elliptic problems. Proc. Indian Acad. Sci. (Math Sci.), Springer 117(1), 109-145 (2007).

[14] G. Karniadakis and S.J. Spencer: Spectral/*hp* element methods for CFD. Oxford University Press (1999).

[15] V.A. Kondratiev: The smoothness of the solution of Dirichlet's problems for second order elliptic equations in a region with a piecewise smooth boundary. Differential nye Uraneniya, 6(10), 1831-1843 (Differential Equations, 6, 1392-1401) (1970).

[16] J.E. Osborn: Spectral approximation for compact operators. Math. Comp. 29, 712-725 (1975).

[17] Ch. Schwab: *p* and *h-p* finite element methods, Clarendon Press, Oxford (1998).

[18] S. K. Tomar: *h-p* spectral element methods for elliptic problems on non-smooth domains using parallel computers, Ph.D. Thesis (IIT Kanpur, INDIA) (2001); Reprint available as Tec. Rep. no. 1631, Department of Applied Mathematics, University of Twente, The Netherlands. http://www.math.utwente.nl/publications .